**The field axioms**

**Closure of F under addition and multiplication**

For all a, b in F, both a + b and a · b are in F (or more formally, + and · are binary operations on F).

**Associativity of addition and multiplication**

For all a, b, and c in F, the following equalities hold: a + (b + c) = (a + b) + c and a · (b · c) = (a · b) · c.

**Commutativity of addition and multiplication**

For all a and b in F, the following equalities hold: a + b = b + a and a · b = b · a.

**Existence of additive and multiplicative identity elements**

There exists an element of F, called the additive identity element and denoted by 0, such that for all a in F, a + 0 = a. Likewise, there is an element, called the multiplicative identity element and denoted by 1, such that for all a in F, a · 1 = a. To exclude the trivial ring, the additive identity and the multiplicative identity are required to be distinct.

**Existence of additive inverses and multiplicative inverses**

For every a in F, there exists an element −a in F, such that a + (−a) = 0. Similarly, for any a in F other than 0, there exists an element a−1 in F, such that a · a−1 = 1. (The elements a + (−b) and a · b−1 are also denoted a − b and a/b, respectively.) In other words, subtraction and division operations exist.

**Distributivity of multiplication over addition**

For all a, b and c in F, the following equality holds: a · (b + c) = (a · b) + (a · c).

**Proposition 1.14 from Rudin.** The axioms for addition imply the following statements.

1. If then (cancellation)
2. If then (uniqueness of the additive identity)
3. If then (uniqueness of the additive inverse)
4. (double negation)

*Proof*. If , the axioms for addition give

This proves (a). Take in (a) to obtain (b). Take in (a) to obtain (c). Since , (c) (with in place of ) gives (d).

**Proposition 1.15 from Rudin**. The axioms for multiplication imply the following statements.

1. If and then (cancellation)
2. If and then (uniqueness of the multiplicative identity)
3. If and then (uniqueness of the multiplicative inverse)
4. If then .

**Proposition 1.16 from Rudin**. The field axioms imply the following statements, for any , , and .

1. .
2. If and then .
3. .

*Proof*. . Hence 1.14(b) implies that , and (a) holds. Next, assume , , but . Then (a) gives

a contradiction. Thus (b) holds. The first equality in (c) comes from

combined with 1.14(c); the other half of (c) is proved in the same way. Finally,

by (c) and 1.14(d).